

Symmetry of Quantum Torus with Crossed Product Algebra

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ABSTRACT

In this paper, we study the symmetry of quantum torus with the concept of crossed product algebra. As a classical counterpart, we consider the orbifold of classical torus with complex structure and investigate the transformation property of classical theta function. An invariant function under the group action is constructed as a variant of the classical theta function. Then our main issue, the crossed product algebra representation of quantum torus with complex structure under the symplectic group is analyzed as a quantum version of orbifolding. We perform this analysis with Manin's so-called model II quantum theta function approach. The symplectic group $Sp(2n, \mathbb{Z})$ satisfies the consistency condition of crossed product algebra representation. However, only a subgroup of $Sp(2n, \mathbb{Z})$ satisfies the consistency condition for orbifolding of quantum torus.

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1. Introduction

Classical theta functions [1] can be regarded as state functions on classical tori, and have played an important role in the string loop calculation [2, 3]. Recently, Manin [4, 5, 6] introduced the concept of quantum theta function as a quantum counterpart of classical theta function. In our previous work [7], we clarified the relationship between Manin's quantum theta function and the theta vector [8, 9, 10] which Schwarz introduced earlier. In [7], we showed the connection between the classical theta function and the so-called kq representation which appeared in the physics literature [11, 12]. Then we showed that the Manin's quantum theta function corresponds to the quantum version of the kq representation. In the physics literature, quantum theta functions are related with noncommutative solitons [13] whose solutions are given in terms of projection operators [14, 13, 15]. Under the lattice translation, quantum theta function maintains the transformation property of classical theta function. Manin's construction of quantum theta function [5, 6] is based on the algebra valued inner product of the theta vector, and this construction is a generalization of Boca's construction of projection operators on the \mathbb{Z}_4 -orbifold of noncommutative two torus [16].

In the algebra valued inner product one can make the inner product of the dual algebra, the representation of the perpendicular lattice space, be invertible or proportional to the identity operator. This makes the algebra valued inner product be a projection operator [17]. In Boca's work [16], the projection operators on the \mathbb{Z}_4 -orbifold of noncommutative two torus were constructed based on the algebra valued inner product that Rieffel [17] used in his classic work on projective modules over noncommutative tori.

One can consider a symmetry group defining an orbifold from the view point of the crossed product algebra of the original algebra with the given symmetry group [18, 19, 13]. Therefore in order to find a representation of an orbifold algebra, one has to find a representation of the group compatible to that of the original algebra. In the Boca's work, the action of \mathbb{Z}_4 -quotient was represented as the Fourier transformation, and the algebra valued inner product was evaluated with the eigenstates of Fourier transformation [16].

When the consistency conditions for the representation of crossed product algebra are fulfilled, the group of the crossed product algebra behaves as a symmetry group of the

original algebra. The consistency conditions for crossed product algebra are basically having compatible actions of the group acting on the original algebra and on the module.

For quantum tori, there are two types of symmetries. One is a symmetry under a group action, and the other is a symmetry under deformation of the algebra, the so-called Morita equivalence [20]. Here, we restrict our discussion to the symmetry under a group action that is not related to the Morita equivalence.

In this paper, we first consider classical functions under orbifolding of torus and try to find an invariant function under the symplectic group $Sp(2n, \mathbb{Z})$. We then look into the representation of crossed product algebra as a way of orbifolding in the quantum (noncommutative) case.

The organization of the paper is as follows. In section 2, we review orbifolding of classical torus and construct an invariant function under the action of $Sp(2n, \mathbb{Z})$. In section 3, we first review the crossed product algebra and its consistency conditions. Then, we check the consistency conditions of our crossed product algebra with the group $Sp(2n, \mathbb{Z})$ via the approach of Manin's model II quantum theta function. In section 4, we conclude with discussion.

2. Orbifolding and classical theta function

In this section, we first consider orbifolding under a group action. A classical function f on an orbifold $X = M/G$ should satisfy

$$f(g \cdot x) = f(x), \quad \forall g \in G, \quad x \in M. \quad (1)$$

Now, we consider the case in which M is a complex torus. Let $M = \mathbb{C}^n / \Lambda$ ($\Lambda \cong \mathbb{Z}^{2n}$) be a complex torus. If M can be embedded in a projective space $\mathbb{C}P^N$ for some N , then it is called an abelian variety. For M to be an abelian variety, there must exist a polarization, a positive line bundle on M . A positive line bundle L on M should satisfy that $\int_C c_1(L) > 0$, for any curve C in M , where $c_1(L)$ is the first Chern class of L as an element of $H^2(M, \mathbb{Z}) \cap H^{1,1}(M, \mathbb{R})$. Explicitly, $c_1(L) = \sum \delta_\alpha dx_\alpha \wedge dy_\alpha = \sum q_\beta dz_\beta \wedge d\bar{z}_\beta$, $\delta_\alpha \in \mathbb{Z}$, and q_β is pure imaginary. In particular, if $\delta_\alpha = 1$, for all α , then the abelian variety is called *principally polarized*. The moduli space \mathfrak{M} of principally polarized abelian varieties is the

collection of the pair $\{ (M, L) | M = \mathbb{C}^n / \Lambda, L \text{ is a principally polarized line bundle} \}$. Let $\mathbb{H}_n = \{ T | T \in M_n(\mathbb{C}), T^t = T, \text{Im} T > 0 \}$ on which $Sp(2n, \mathbb{Z})$ acts as follows:

$$g \cdot T = (AT + B)(CT + D)^{-1}, \quad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{Z}).$$

Then, $\mathfrak{M} = Sp(2n, \mathbb{Z}) \backslash \mathbb{H}_n$.

Now, we consider an action of a group G on M . In other words, a map from $G \times M$ to M , such that for every $g \in G$, g is an automorphism of M preserving complex structure T and the group structure. Then, g induces a linear map from \mathbb{C}^n to \mathbb{C}^n sending Λ to Λ . It means that g belongs to $GL(n, \mathbb{C})$ and also $GL(2n, \mathbb{Z})$ which is given in terms of the basis of $\Lambda (\cong \mathbb{Z}^{2n})$, whose determinant is ± 1 . Additionally, if we impose that g preserves L , then g preserves $c_1(L)$, so that

$$c_1(L) = \sum dx_\alpha \wedge dy_\alpha = g^*(c_1(L)) = \sum d(g^*x_\alpha) \wedge d(g^*y_\alpha).$$

It implies that $g \in Sp(2n, \mathbb{Z})$. Then we can define an orbifold M/G with the preserved polarization L .

If $g \in GL(n, \mathbb{C})$ and $g \in Sp(2n, \mathbb{Z})$, then $T' = g \cdot T = T$ as we see below. Hence, only a subgroup of $Sp(2n, \mathbb{Z})$, namely $GL(n, \mathbb{C}) \cap Sp(2n, \mathbb{Z})$, acts as a symmetry group for orbifolding.

For $g \in Sp(2n, \mathbb{Z})$, it acts on the basis as follows:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} T \\ I \end{pmatrix} = \begin{pmatrix} AT + B \\ CT + D \end{pmatrix} \sim \begin{pmatrix} (AT + B)(CT + D)^{-1} \\ I \end{pmatrix} = \begin{pmatrix} T' \\ I \end{pmatrix}.$$

On the other hand, for $g \in GL(n, \mathbb{C})$ it acts as follows:

$$\begin{pmatrix} T \\ I \end{pmatrix} \cdot g^t = \begin{pmatrix} T \cdot g^t \\ I \cdot g^t \end{pmatrix} \sim \begin{pmatrix} T \cdot g^t \cdot g^{-t} \\ I \end{pmatrix} = \begin{pmatrix} T \\ I \end{pmatrix}.$$

Since the two actions should yield the same result, we get to the result that $T' = g \cdot T = T$.

We now consider whether the classical theta function θ is well defined on the above mentioned orbifold. The classical theta function θ is a complex valued function on \mathbb{C}^n satisfying the following relation.

$$\theta(z + \lambda') = \theta(z) \quad \text{for } z \in \mathbb{C}^n, \lambda' \in \Lambda', \quad (2)$$

$$\theta(z + \lambda) = c(\lambda) e^{q(\lambda, z)} \theta(z) \quad \text{for } \lambda \in \Lambda, \quad (3)$$

where $\Lambda' \oplus \Lambda \subset \mathbb{C}^n$ is a discrete sublattice of rank $2n$ split into the sum of two sublattices of rank n , isomorphic to \mathbb{Z}^n , and $c : \Lambda \rightarrow \mathbb{C}$ is a map and $q : \Lambda \times \mathbb{C} \rightarrow \mathbb{C}$ is a biadditive pairing linear in z .

The above property reflects the fact that the classical theta function lives on \mathbb{C}^n not on \mathbb{T}^{2n} . The function $\theta(z, T)$ satisfying (2) and (3) can be defined as

$$\theta(z, T) = \sum_{k \in \mathbb{Z}^n} e^{\pi i(k^t T k + 2k^t z)} \quad (4)$$

where $T \in \mathbb{H}_n$. With the above definition, $c(\lambda)$ and $q(\lambda, z)$ in (3) are given explicitly by $c(\lambda) = e^{-\pi i m^t T m}$ and $q(\lambda, z) = -2\pi i m^t z$ when $\lambda = Tm$, $m \in \mathbb{Z}^n$. Also $z \in \mathbb{C}^n$ transforms as

$$g \cdot z = z' = (CT + D)^{-t} z, \quad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{Z}), \quad (5)$$

where “ $-t$ ” denotes the transposed inverse. Under this modular transformation, the classical theta function transforms as follows.

$$g \cdot \theta(z, T) = \theta(z', T') = \xi_g \det(CT + D)^{\frac{1}{2}} e^{\pi i \{z^t (CT + D)^{-1} C z\}} \theta(z, T), \quad \forall g \in Sp(2n, \mathbb{Z}) \quad (6)$$

where ξ_g is an eighth root of unity depending on the group element g [1].

Now, we like to find a compatible function on the orbifold in which the complex structure is preserved, $g \cdot T = T$. For this, we first try to construct a new function which has the symmetry properties of the classical theta function, (2) and (3). We define a new function as a linear combination of the classical theta functions under the group action:

$$\Theta_1(z, T) = \sum_{g \in G} g \cdot \theta(z, T). \quad (7)$$

Clearly the above function is invariant under the group action,

$$h \cdot \Theta_1(z, T) = \sum_{g \in G} h \cdot g \cdot \theta(z, T) = \sum_{g' \in G} g' \cdot \theta(z, T) = \Theta_1(z, T), \quad \forall h \in G. \quad (8)$$

However, this function does not possess the symmetry properties of the classical theta function (2) and (3). This is because the condition (2) is not satisfied by $\Theta_1(z, T)$, since

$$g \cdot \theta(z + \lambda', T) = \theta(g \cdot (z + \lambda'), g \cdot T) = \theta(g \cdot z + g \cdot \lambda', T) \neq \theta(g \cdot z, T), \quad (9)$$

where $g \cdot \lambda' \in \Lambda + \Lambda'$ for some $\lambda' \in \Lambda'$ due to the modular transformation $g \cdot \lambda' = (CT + D)^{-t} \lambda'$. For the condition (3), each $g \cdot \theta$ in $\Theta_1(z, T)$ in (7) gets a different factor for a lattice shift in Λ :

$$\begin{aligned} g \cdot \theta(z + \lambda, T) &= \theta(g \cdot (z + \lambda), g \cdot T) = \theta(g \cdot z + g \cdot \lambda, T) \\ &\neq \theta(g \cdot z + \lambda, T) \text{ for } \lambda \in \Lambda, \end{aligned} \quad (10)$$

since again $g \cdot \lambda = (CT + D)^{-t} \lambda \neq \lambda$ and belongs to $\Lambda + \Lambda'$ in general. Thus the function $\Theta_1(z, T)$ fails to preserve the transformation properties of the classical theta function, (2) and (3), though it is a well defined function on the orbifold.

In (4), the above result was due to the product $k^t z$ in the exponent. So we need to find a new combination of this type of product under the modular transformation that preserves the complex structure. Since a symplectic product preserves the complex structures, we modify the classical theta function as follows.

$$\tilde{\Theta}(z, T) = \sum_{\underline{k}} \exp(-\pi H_T(\underline{k}, \underline{k}) + 2\pi i \operatorname{Im}[H_T(\underline{k}, z)]) \quad (11)$$

where

$$H_T(s, z) \equiv s^t (\operatorname{Im} T)^{-1} z^* \text{ for } s, z \in \mathbb{C}^n. \quad (12)$$

Here, T is the complex structure given before, and \underline{k} denotes the lattice point given by $\underline{k} = Tk_1 + k_2$ with $k_1, k_2 \in \mathbb{Z}^n$, and $z \in \mathbb{C}^n$ is given as usual with $z = Tx_1 + x_2$ with $x_1, x_2 \in \mathbb{R}^n$. Here, we notice that $\operatorname{Im}[H_T(\underline{k}, z)] = \operatorname{Im}[\underline{k}^t (\operatorname{Im} T)^{-1} z^*] = k_1^t x_2 - k_2^t x_1$. If we denote \underline{x} as $z = Tx_1 + x_2 \equiv \underline{x}$ and the same for $\underline{y} = Ty_1 + y_2$ with $y_1, y_2 \in \mathbb{R}^n$, then $H_T(\underline{x}, \underline{y}) = \underline{x}^t (\operatorname{Im} T)^{-1} \underline{y}^*$ is an invariant combination under the modular transformation, $T' = (AT + B)(CT + D)^{-1}$, $\underline{x}' = (CT + D)^{-t} \underline{x}$ and the same for \underline{y} , for any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{Z})$. One can check that the above transformation of the complex coordinate \underline{x} is compatible with the following coordinate transformation in the real basis.

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (13)$$

The first term in the exponent in (11) is invariant under the modular transformation as we shall see in the next section, and the second term is also invariant since it is a symplectic product that preserves the complex structure. Thus, our modified theta function $\tilde{\Theta}$ is invariant under the modular transformation, and it is a well defined function on the above orbifold.

In fact, we can view this as follows. The classical theta function θ in (4) is summed over only one of the two \mathbb{Z}^n lattices Λ, Λ' in the $2n$ -torus. Our modified theta function $\tilde{\Theta}$ is summed over the both lattices, and its property under lattice translation is changed from that of the classical theta function. The new function $\tilde{\Theta}$ is invariant under the lattice translation in both directions, Λ and Λ' . And this property is preserved under the group action.

In general, for a manifold M on which a group G is acting, one can define invariant functions on M under the action of the group G as the functions on the orbifold M/G . In the next section, we will do a quantum counterpart of the above analysis with crossed product algebra.

3. Quantum torus with crossed product algebra

In order to consider an orbifolding of quantum torus, we have to express the group action in terms of the representation of the crossed product algebra. So, in this section we first review briefly about the crossed product algebra and its representation, then we will investigate the representation of crossed product algebra for orbifolding.

3.1 Crossed product algebra

We now consider the crossed product algebra and its representation [18, 13].

Let G , a group, act on an algebra \mathcal{A} . More explicitly there is a group homomorphism

$$\varepsilon : G \rightarrow \text{Aut}(\mathcal{A}).$$

Then we define the crossed product algebra $\mathcal{B} = \mathcal{A} \rtimes_{\varepsilon} G$, which is $\mathcal{A}[G] = \{b \mid b : G \rightarrow \mathcal{A}\}$ as a set. And we formally express $b \in \mathcal{B}$ as $\sum_{g \in G} b_g g$, where $b_g = b(g) \in \mathcal{A}$. Here, addition

and scalar product are defined naturally. To define multiplication we require the following relation:

$$g \cdot b_{g'} g^{-1} = \varepsilon(g)(b_{g'}), \quad g, g' \in G, \quad b_{g'} \in \mathcal{A}. \quad (14)$$

For $b, c, d \in \mathcal{B}$ with $b = \sum_{g \in G} b_g g$, $c = \sum_{g' \in G} c_{g'} g'$, $d = \sum_{h \in G} d_h h$, we can express the multiplication $b *_{\varepsilon} c = d$ as

$$\begin{aligned} b *_{\varepsilon} c &= \sum_g b_g g \cdot \sum_{g'} c_{g'} g' = \sum_{g, g'} b_g g \cdot c_{g'} g' \\ &= \sum_{g, h} b_g \varepsilon(g)(c_{g^{-1}h}) h \\ &= \sum_h d_h h = d, \end{aligned} \quad (15)$$

where we set $g' = g^{-1}h$, $d_h = b_g \varepsilon(g)(c_{g^{-1}h})$, and used the relation (14).

If there are representations π , u which are a representation of \mathcal{A} and a representation of the group G , respectively, on a module \mathcal{H} ,

$$\pi : \mathcal{A} \rightarrow \text{End}(\mathcal{H}), \quad u : G \rightarrow \text{Aut}(\mathcal{H}),$$

then (14) leads to the following condition for any representation of the crossed product algebra should satisfy:

$$u(g)\pi(a)u(g^{-1}) = \pi(\varepsilon(g)(a)), \quad \forall a \in \mathcal{A}, \quad \forall g \in G. \quad (16)$$

Furthermore, if there exists an \mathcal{A} valued inner product ${}_{\mathcal{A}} \ll, \gg$ on \mathcal{H} , then the following should be also satisfied for consistency [18],

$$\varepsilon(g)({}_{\mathcal{A}} \ll \xi, \eta \gg) = {}_{\mathcal{A}} \ll u(g)\xi, u(g)\eta \gg, \quad \text{for } g \in G, \quad \xi, \eta \in \mathcal{H}. \quad (17)$$

Here, ${}_{\mathcal{A}} \ll \xi, \eta \gg$ denotes the \mathcal{A} -algebra valued inner product to be defined below, which belongs to \mathcal{A} . We changed the notation for the algebra valued inner product from the single bracket in our previous work [7] to the double bracket to distinguish it from the usual scalar product which we will denote with the single bracket below.

3.2 Symmetry transformation

In [6], Manin constructed the quantum theta function in two ways which he called model I and model II. The model I basically follows the Rieffel's way of constructing projective modules over noncommutative tori. Thus in the model I, one deals with Schwartz functions on \mathbb{R}^n for complex n -torus. And the scalar product is defined as

$$\langle \xi, \eta \rangle = \int \xi(x_1) \overline{\eta(x_1)} d\mu(x_1), \quad x_1 \in \mathbb{R}^n \quad (18)$$

where $\overline{\eta(x_1)}$ denotes the complex conjugation of $\eta(x_1)$, and $d\mu(x_1)$ denotes the Haar measure in which \mathbb{Z}^n has covolume 1.

In the model II, one deals with holomorphic functions on \mathbb{C}^n , and the scalar product is defined as

$$\langle \xi, \eta \rangle_T = \int_{\mathbb{C}^n} \xi(\underline{x}) \overline{\eta(\underline{x})} e^{-\pi H_T(\underline{x}, \underline{x})} d\nu \quad (19)$$

where $d\nu$ is the translation invariant measure making \mathbb{Z}^{2n} a lattice of covolume 1 in \mathbb{R}^{2n} . Here, $\underline{x} = Tx_1 + x_2$ with $x_1, x_2 \in \mathbb{R}^n$. The complex structure T is given by an $n \times n$ complex valued matrix, and $H_T(\underline{x}, \underline{x}) = \underline{x}^t (\text{Im} T)^{-1} \underline{x}^*$ as in (12).

Now, we do the analysis with the model II quantum theta function. For consistency of the representation of a crossed product algebra $\mathcal{B} = \mathcal{A} \rtimes G$, we need to define the following as explained in the previous subsection :

- (I) $\pi : \mathcal{A} \rightarrow \text{End}(\mathcal{H})$
- (II) $u : G \rightarrow \text{Aut}(\mathcal{H})$
- (III) $\varepsilon : G \rightarrow \text{Aut}(\mathcal{A})$, such that $u(g)\pi(a)u(g^{-1}) = \pi(\varepsilon(g)(a))$
- (IV) $\ll, \gg : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$, such that $\varepsilon(g) \ll f, h \gg = \ll u(g)f, u(g)h \gg$.

Let M be any locally compact Abelian group and \widehat{M} be its dual group and define $\mathcal{G} \equiv M \times \widehat{M}$. And, let π be a representation of \mathcal{G} on $L^2(M)$ such that

$$\pi_x \pi_y = \alpha(x, y) \pi_{x+y} = \alpha(x, y) \overline{\alpha}(y, x) \pi_y \pi_x \quad \text{for } x, y \in \mathcal{G} \quad (20)$$

where α is a map $\alpha : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C}^*$ satisfying

$$\alpha(x, y) = \alpha(y, x)^{-1}, \quad \alpha(x_1 + x_2, y) = \alpha(x_1, y) \alpha(x_2, y).$$

We also define $S(D)$ as the space of Schwartz functions on D which we take as a discrete subgroup of \mathcal{G} . For $\Phi \in S(D)$, it can be expressed as $\Phi = \sum_{w \in D} \Phi(w) e_{D,\alpha}(w)$ where $e_{D,\alpha}(w)$ is a delta function with support at w and obeys the following relation.

$$e_{D,\alpha}(w_1) e_{D,\alpha}(w_2) = \alpha(w_1, w_2) e_{D,\alpha}(w_1 + w_2). \quad (21)$$

From now on, we take M as \mathbb{R}^n . Let \mathcal{A} be $S(D)$ valued functions on \mathbb{H}_n . More explicitly

$$\mathcal{A} = S(D) \otimes \mathcal{F}(\mathbb{H}_n) = \{a \mid a : \mathbb{H}_n \rightarrow S(D)\}, \quad (22)$$

where $\mathcal{F}(\mathbb{H}_n)$ is an algebra of smooth complex functions on \mathbb{H}_n . Then $a(T) = \sum_{w \in D} a_{T,w} e(w)$, where $a_{T,w} \in \mathbb{C}$. Let \mathcal{H} be given as follows.

$$\begin{aligned} \mathcal{H} = \{f \mid f : \mathbb{R}^n \times \widehat{\mathbb{R}}^n \times \mathbb{H}_n \rightarrow \mathbb{C}, \\ < f(x, T), f(x, T) >_T = \int |f(x, T)|^2 e^{-\pi H_T(x, x)} dx < \infty, \forall T\} \end{aligned} \quad (23)$$

where $x \in \mathbb{R}^n \times \widehat{\mathbb{R}}^n$, $T \in \mathbb{H}_n$ and from here on $H_T(x, y)$ that we used above denotes $H_T(\underline{x}, \underline{y})$ defined in the section 2 for notational convenience. In other words, \mathcal{H} are global sections of \mathbb{H} , a vector bundle over \mathbb{H}_n , where the fiber over T is

$$\mathbb{H}_T = \{\xi \mid \xi : \mathbb{R}^n \times \widehat{\mathbb{R}}^n \rightarrow \mathbb{C}, \quad < \xi, \xi >_T < \infty\}. \quad (24)$$

Let the group G be $Sp(2n, \mathbb{Z})$ and we now carry out the steps (I) through (IV) that we listed above.

(I) Before we define π , we need to define a map π_0 from $S(D)$ to $End(\mathcal{H})$:

$$\pi_0 : e(w) \rightarrow \pi_w \quad \text{for } w \in D$$

where

$$(\pi_w f)(x, T) = e^{-\pi H_T(x, w) - \frac{\pi}{2} H_T(w, w)} f(x + w, T). \quad (25)$$

Let $a \in \mathcal{A}$, where $a(T) = \sum_w a_{T,w} e(w)$. Now, we define π as follows.

$$(\pi(a)f)(x, T) = [\pi_0(a(T))f](x, T). \quad (26)$$

(II) We define u as follows.

$$(u(g)f)(x, T) = f(g \cdot x, g \cdot T), \quad (27)$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{Z})$, $g \cdot x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-t} x$, and $g \cdot T = (AT + B)(CT + D)^{-1}$.

For the remaining steps we need to use the following two lemmas.

Lemma 1 :

$$H_T(x, y) = H_{g \cdot T}(g \cdot x, g \cdot y). \quad (28)$$

Lemma 2 :

$$\langle f, h \rangle_{g \cdot T} = \langle u(g)f, u(g)h \rangle_T. \quad (29)$$

Proof of the lemma 1:

We first want to show that

$$Im(g \cdot T) = Im((AT + B)(CT + D)^{-1}) = (C\bar{T} + D)^{-t} Im(T)(CT + D)^{-1}. \quad (30)$$

Then the proof of the lemma 1 is given by the following steps.

$$\begin{aligned} H_{g \cdot T}(g \cdot x, g \cdot y) &= ((CT + D)^{-t} \underline{x})^t (Im(g \cdot T))^{-1} ((CT + D)^{-t} \underline{y})^* \\ &= \underline{x}^t (CT + D)^{-1} (CT + D) (Im(T))^{-1} (C\bar{T} + D)^t (C\bar{T} + D)^{-t} \underline{y}^* \\ &= \underline{x}^t (Im(T))^{-1} \underline{y}^* = H_T(x, y). \end{aligned}$$

Thus, we only have to show (30). We can prove it with the three generators of $Sp(2n, \mathbb{Z})$ [1].

$$i) \quad g = \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix}, \quad A \in GL(n, \mathbb{Z}) \quad (31)$$

$$ii) \quad g = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, \quad B^t = B, \quad B \in gl(n, \mathbb{Z}) \quad (32)$$

$$iii) \quad g = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \quad (33)$$

For the first two cases, (30) can be shown trivially. For the case iii), we need to show the following:

$$Im T' = \bar{T}^{-t} (Im T) T^{-1} = \bar{T}^{-1} (Im T) T^{-1} \quad (34)$$

where $T' = g \cdot T = -T^{-1}$.

Now, we prove (34).

Let $T = T_1 + iT_2$ and $T' = T'_1 + iT'_2$. Then from $T'T = -I$, we get $T'_1T_1 - T'_2T_2 = -I$ and $T'_2T_1 + T'_1T_2 = 0$. Then the statement we want to prove becomes $T'_2 = \overline{T}^{-1}T_2T^{-1}$, or equivalently,

$$\overline{T}T'_2T = T_2. \quad (35)$$

The left hand side of (35) is

$$\begin{aligned} L.H.S. &= (T_1 - iT_2)T'_2(T_1 + iT_2) \\ &= (T_1T'_2T_1 + T_2T'_2T_2) + i(-T_2T'_2T_1 + T_1T'_2T_2). \end{aligned}$$

Using $T'_1T_1 - T'_2T_2 = -I$ and $T'_2T_1 + T'_1T_2 = 0$ together with the property that T_i, T'_i are symmetric, then we can easily show that

$$L.H.S. = T_2 = R.H.S.$$

Proof of the lemma 2:

The left hand side of (29) is

$$\begin{aligned} L.H.S. &= \langle f, h \rangle_{g \cdot T} \\ &= \int f(x, g \cdot T) \overline{h(x, g \cdot T)} e^{-\pi H_{g \cdot T}(x, x)} dx, \end{aligned}$$

and the right hand side of (29) is

$$\begin{aligned} R.H.S. &= \langle u(g)f, u(g)h \rangle_T \\ &= \int f(g \cdot x, g \cdot T) \overline{h(g \cdot x, g \cdot T)} e^{-\pi H_T(x, x)} dx \\ &= \int f(x, g \cdot T) \overline{h(x, g \cdot T)} e^{-\pi H_{g \cdot T}(x, x)} dx, \end{aligned}$$

where we used the lemma 1 in the final step.

(III) We define $\varepsilon : G \rightarrow \text{Aut}(\mathcal{A})$ such that $u(g)\pi(a)u(g^{-1}) = \pi(\varepsilon(g)(a))$.

Let $a(T)$ be $\sum a_{T,w}e(w)$. The left hand side can be evaluated as follows.

$$\begin{aligned} (u(g)\pi(a)u(g^{-1})f)(x, T) &= (\pi(a)u(g^{-1})f)(g \cdot x, g \cdot T) \\ &= \sum_w a_{g \cdot T, w} e^{-\pi H_{g \cdot T}(g \cdot x, w) - \frac{\pi}{2} H_{g \cdot T}(w, w)} f(x + g^{-1} \cdot w, T) \end{aligned}$$

If we define $\varepsilon(g)(a)(T) = \sum_w a_{g \cdot T, w} e(g^{-1} \cdot w)$, then the right hand side is given by

$$\begin{aligned} \pi(\varepsilon(g)(a)f)(x, T) &= \sum_w a_{g \cdot T, w} \pi(g^{-1} \cdot w) f(x, T) \\ &= \sum_w a_{g \cdot T, w} e^{-\pi H_{g \cdot T}(g \cdot x, w) - \frac{\pi}{2} H_{g \cdot T}(w, w)} f(x + g^{-1} \cdot w, T). \end{aligned}$$

In the last equality we used the lemma 1. So those two sides are equal. Using the lemma 1, one can also show the following.

$$u(g)\pi_w u(g^{-1}) = \varepsilon(g)\pi_w = \pi_{g^{-1} \cdot w}. \quad (36)$$

(IV) We define an \mathcal{A} -valued inner product on \mathcal{H} as follows.

$$\ll f, h \gg (T) = \sum_w \langle f, \pi_w h \rangle_T e(w) \quad (37)$$

where $\langle f, \pi_w(h) \rangle_T = \langle f(x, T), \pi_w h(x, T) \rangle_T$.

In other words if $a = \ll f, h \gg$ then $a_{T,w} = \langle f, \pi_w h \rangle_T$.

Now, we want to check that $\varepsilon(g) \ll f, h \gg = \ll u(g)f, u(g)h \gg$ holds.

Recall that

$$\varepsilon(g)(a)(T) = \sum_w a_{g \cdot T, w} e(g^{-1} \cdot w).$$

The left hand side is given by

$$\begin{aligned} (\varepsilon(g)(\ll f, h \gg))(T) &= \sum_w \langle f, \pi_w h \rangle_{g \cdot T} e(g^{-1} \cdot w) \\ &= \sum_w \langle f, \pi_{g \cdot w} h \rangle_{g \cdot T} e(w). \end{aligned}$$

The right hand side is given by

$$\begin{aligned}
\ll u(g)f, u(g)h \gg_T &= \sum_w \langle u(g)f, \pi_w u(g)h \rangle_T e(w) \\
&= \sum_w \langle f, u(g)^{-1} \pi_w u(g)h \rangle_{g \cdot T} e(w) \\
&= \sum_w \langle f, \pi_{g \cdot w} h \rangle_{g \cdot T} e(w),
\end{aligned}$$

where we used the lemma 2 and (36).

3.3 Orbifolding quantum torus

We consider an orbifolding of quantum torus with a polarized complex structure T . The symmetry group preserving the polarized complex structure is the subgroup $G_T = \{g \in Sp(2n, \mathbb{Z}) | g \cdot T = T\}$ of $Sp(2n, \mathbb{Z})$. Orbifolding the quantum torus with a complex structure T corresponds to the crossed product algebra discussed in the previous section with fixed T .

Let $A_T = S(D)$ and $\mathbb{H}_T = \{f_T | f_T : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}, \|f_T\|^2 = \int |f_T(x)|^2 e^{-\pi H_T(x,x)} dx < \infty\}$. Now, we can define the crossed product algebra, $A_T \rtimes G_T$, naturally from the construction in the section 3.2:

1. $\pi_T : A_T \rightarrow \text{End}(\mathbb{H}_T)$
2. $u_T : G_T \rightarrow \text{Aut}(\mathbb{H}_T)$
3. $\varepsilon_T : G_T \rightarrow \text{Aut}(A_T)$ such that $u_T(g)\pi_T(a)u_T(g^{-1}) = \pi_T(\varepsilon_T(g)(a))$
4. $\ll, \gg_T : \mathbb{H}_T \times \mathbb{H}_T \rightarrow A_T$ such that $\varepsilon_T(g) \ll f_T, h_T \gg_T = \ll u_T(g)f_T, u_T(g)h_T \gg_T$.

Here, $\pi_T, u_T, \varepsilon_T, \ll, \gg_T, f_T$ satisfy the following relations:

$$\begin{aligned}
(\pi_T(a(T))f_T)(x) &= (\pi(a)f)(x, T), \\
(u_T(g)f_T)(x) &= (u(g)f)(x, T), \\
(\varepsilon(g)(a))(T) &= \varepsilon_T(g)(a(T)), \\
\ll f_T, h_T \gg_T &= \ll f, h \gg(T),
\end{aligned}$$

where $f_T(x) = f(x, T)$, $a \in S(D) \otimes \mathcal{F}(\mathbb{H}_n)$ and $g \in G_T$. If we choose $f(x, T) = 1$, then $\varepsilon(g) \ll 1, 1 \gg = \ll u(g)1, u(g)1 \gg = \ll 1, 1 \gg$, and thus $\ll 1, 1 \gg$ which belongs to the algebra \mathcal{A} is $Sp(2n, \mathbb{Z})$ invariant. Since $\ll 1, 1 \gg (T) = \sum_{w \in D} e^{-\frac{\pi}{2} H_T(w, w)} e(w)$ is the Manin's model II quantum theta function, this also tells us that the model II quantum theta function is well defined on the orbifolds of quantum complex torus. We further notice that Boca's projection operator [16] on the $\mathbb{Z}/4\mathbb{Z}$ orbifold of quantum 2-torus with $T = i$ corresponds to a special case of this construction.

4. Conclusion

In this paper, we investigate the symmetry of quantum torus with the group $Sp(2n, \mathbb{Z})$.

First, we investigate the orbifolding of classical complex torus. It turns out that the orbifold group for complex n -torus leaving the complex structure and its polarization intact is a subgroup of $Sp(2n, \mathbb{Z})$. Also, the classical theta function is not invariant under the $Sp(2n, \mathbb{Z})$ transformation, and we construct a variant of the classical theta function as an invariant function under the transformation of $Sp(2n, \mathbb{Z})$. Then as a quantum counterpart, we investigate the representation of crossed product algebra of quantum torus with $Sp(2n, \mathbb{Z})$ via Manin's model II quantum theta function approach.

In the Manin's model I approach, the dimension of the Hilbert space variable x_1 , which is n for quantum \mathbb{T}^{2n} , does not match the dimension of the fundamental representation of $Sp(2n, \mathbb{Z})$, which is $2n$. On the other hand, in the model II case the dimension of the Hilbert space variable $x = (x_1, x_2)$ exactly matches that of the group. Therefore in the model I case the group action cannot act directly on the variables of the Hilbert space. Thus one has to devise a transformation such as Fourier transformation as in the Boca's work [16], where \mathbb{Z}_4 acts directly on the functions as a Fourier transformation, not on the variables of the functions. This type of difficulty comes from the fact that in the model I case the number of variables of the functions is half of that of the phase space as it is typical in the conventional quantization. In the model II approach, the above mentioned difficulty does not arise. The group action can be defined nicely on the module as it acts on the variables.

In conclusion, in the model II case $Sp(2n, \mathbb{Z})$ turns out to be the symmetry group for the quantum torus times \mathbb{H}_n . The orbifolding of quantum torus with complex structure corresponds to the crossed product algebra, $S(D) \rtimes G_T$, where G_T is the subgroup of $Sp(2n, \mathbb{Z})$ fixing the complex structure, $g \cdot T = T$ for $g \in Sp(2n, \mathbb{Z})$. And Manin's model II quantum theta function turns out to be a well defined function over the above orbifold of quantum torus.

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